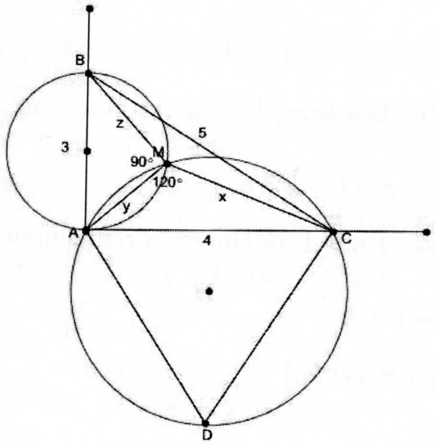
Myagmarsuren Yadamsuren and Béla Kovács

Third solution. Let $S := xy + 2yz + 3zx$.By replacing y with $\sqrt{3}y$ we obtain $S = xy\sqrt{3} + 2yz\sqrt{3} + 3zx$,
where

$$(1)\quad \begin{cases} x^2 + xy\sqrt{3} + y^2 = 25 \\ y^2 + z^2 = 9 \\ z^2 + xz + x^2 = 16 \end{cases} \iff \begin{cases} x^2 + y^2 - 2xy\cos 150^\circ = 25 \\ y^2 + z^2 - 2yz\cos 120^\circ = 9 \\ z^2 + x^2 - 2xz\cos 90^\circ = 16 \end{cases}.$$

System (1) has clear geometric interpretation.



Let ABC be a right triangle with sidelengths $AB = 3, CA = 4, BC = 5$.
Let C_{AB} be a circle with diameter AB.Then any point M which belong to interior ark of $C_{AB} \setminus \{A, B\}$ determine right angle $\angle AMB$.
From the other hand let ADC be equilateral triangle constructed on side AC externally and C_{AC} be circumcircle of triangle ADC.
Then any point $M \in C_{AC} \setminus \{A, C\}$ on the ark of this circle, formed by a chord AC and lying in the same half plain as triangle, determine angle $\angle AMC = 120^\circ$.
Let M be point of intersection of such two arks. Then $\angle AMC = 120^\circ, \angle AMB = 90^\circ$ and, therefore, $\angle BMC = 150^\circ$. Let $y := MA,\ z := MB, x := MC$.
Then x, y, z be positive solution of the system **(1)** and $[AMB] = \dfrac{yz}{2}$,

$$[AMC] = \frac{xy\sin 120^\circ}{2} = \frac{xy}{4}, [BMC] = \frac{xz\sqrt{3}}{4}.$$

Therefore,

$$[ABC] = [AMB] + [AMC] + [BMC] = \frac{yz}{2} + \frac{xy}{4} + \frac{zx\sqrt{3}}{4} =$$

$$= \frac{1}{4\sqrt{3}}\left(2xy\sqrt{3} + xy\sqrt{3} + 3zx\right) = \frac{S}{4\sqrt{3}}.$$

Thus, $S = 4\sqrt{3}\,[ABC] = 4\sqrt{3} \cdot \dfrac{AB \cdot BC}{2} = 4\sqrt{3} \cdot \dfrac{3 \cdot 4}{2} = 24\sqrt{3}.$

Arkady Alt

W2. **(Solution by the proposer.)** This is a (may be hard) refinement of

$$x^2 \leq x^2 - x + 1 - x^2\,(1-x)^4$$

(submitted to the journal "Mathproblems") which in turn is a refinement of

$$x^x \leq x^2 - x + 1$$

(Crux Mathematicorum, problem 3815, Vol. 39(2))
The inequality is equivalent to

$$x \leq \ln\left(x^2 - x + 1 - x\left(1 - x^4\right)\right)$$

Now define $1 - x = h \in [0, 1]$, whence it becomes

$$(1-h)\ln(1-h) \leq \ln\left[1 - h\,(1-h)\left(1+h^3\right)\right]$$

Moreover by tha AGM we get

$$h\,(1-h)\left(1+h^3\right) \leq \frac{(h+1-h)^2}{4}\left(1+h^3\right) \leq \frac{1}{4}2 = \frac{1}{2}$$

so we can use Taylor to write the inequality as

$$-\,(1-h)\sum_{k=1}^{5}\frac{h^k}{k} \leq \sum_{k=1}^{\infty}\frac{h^k\,(1-h)^k\left(1+h^3\right)^k}{k}$$

if and only if

$$\sum_{k=1}^{\infty}\frac{h^k}{k} \geq \sum_{k=1}^{\infty}\frac{h^k\,(1-h)^k\left(1+h^3\right)^k}{k}$$